TOPIC 6: Definite Integrals and Applications of Integration

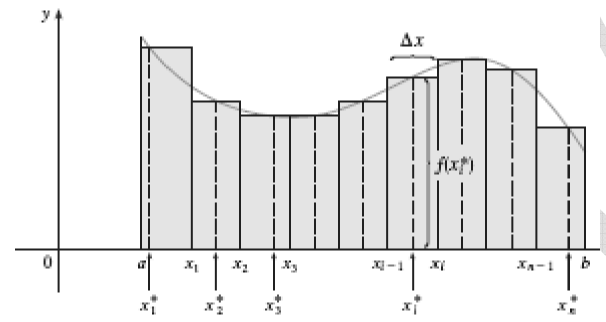
1. DEFINITE INTEGRALS

Definition – Definite Integral
If f is a continuous function defined on the closed interval $[a,b]$, we divide the interval $[a,b]$ into n subintervals $[x_0,x_1],[x_1,x_2],\cdots[x_{n-1},x_n]$ of equal width $\Delta x=\dfrac{b-a}{n}$, with . $a=x_0<x_1<x_2<...<x_n=b$. We choose sample points $x_1^*,x_2^*,\cdots x_n^*$ in these subintervals, so that x_i^* lies in the i th subinterval $[x_{i-1},x_i]$. The **definite integral** of f from a to b is

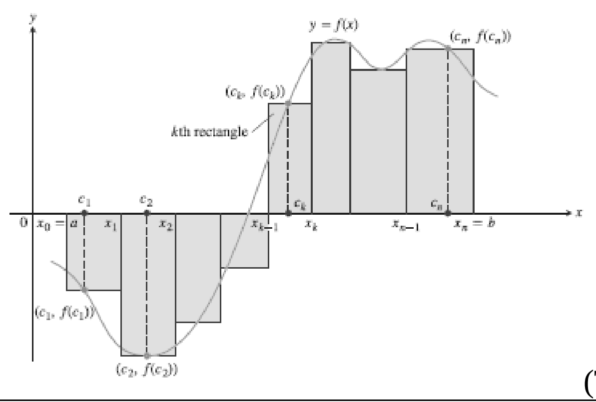
$$\int_a^b f(x)dx=\lim_{\Delta x\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points.

If it does exist, we say that f is **integrable** on [a, b]



$\sum_{i=1}^{n} f(x_i^*)\Delta x$ (S*)



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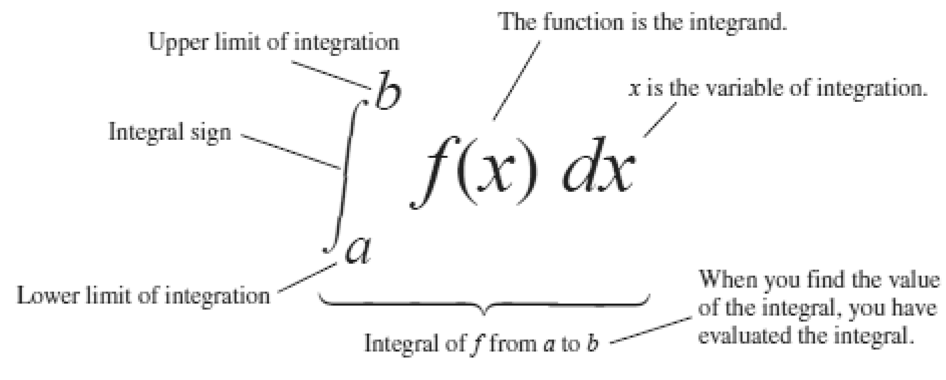

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The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann sum**.

The definite integral $\int_a^b f(x)dx$ is a number; it does not depend on *x*. In fact it can be written using any letter as the variable of integration, and the value of the integral remains unchanged.

Indefinite integral	Definite integral
$\int f(x)dx$ is a function of *x*. It is defined using antiderivatives. $\int f(x)dx, \int f(t)dt$ and $\int f(u)du$ are functions of different variables.	$\int_a^b f(x)dx$ is a number. It is defined as a limit of Riemann sums. *x* is a dummy variable in the sense that $\int_a^b f(x)dx$, $\int_a^b f(t)dt$ and $\int_a^b f(u)du$ are the same number.

Theorem - If f is a continuous function on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is **<u>integrable</u>** on $[a,b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

<u>Properties of the Definite Integrals</u>

Suppose that the definite integrals of f and g from a to b, with $a \le b$, exist, and k is a constant. Then

1. $\int_a^b k\ dx = k(b-a)$
2. Order of limits of integration: $\int_b^a f(x)dx = -\int_a^b f(x)dx$
3. Zero Width Interval: $\int_a^a f(x)dx = 0$
4. Constant Multiple: $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
5. Sum and Difference: $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
6. Additivity: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for $a \le c \le b$
7. Domination: $\int_a^b f(x)dx \ge \int_a^b g(x)dx$, if $f(x) \ge g(x)$ for $a \le x \le b$

   $\int_a^b f(x)dx \ge 0$, if $f(x) \ge 0$ for $a \le x \le b$ (a special case)

<u>Example 1:</u>

Suppose that $\int_{-1}^1 f(x)dx = 5$, $\int_1^4 f(x)dx = -2$ and $\int_{-1}^1 h(x)dx = 7$ then

a) $\int_4^1 f(x)dx = -\int_1^4 f(x)dx = -(-2) = 2$

b) $\int_{-1}^1 [2f(x) + 3h(x)]dx = 2\int_{-1}^1 f(x)dx + 3\int_{-1}^1 h(x)dx = 2(5) + 3(7) = 31$

c) $\int_{-1}^4 f(x)dx = \int_{-1}^1 f(x)dx + \int_1^4 f(x)dx = 5 + (-2) = 3$

The Fundamental Theorem of Calculus establishes a connection between the two branches of calculus: differential calculus and integral calculus. It gives the precise inverse relationship between the derivative and the integral.

The Fundamental Theorem of Calculus, Part 1

If f is a continuous function on $[a,b]$, then the function g defined by

$$g(x)=\int_a^x f(t)dt, \quad a\le x\le b$$

is continuous on $[a,b]$, and differentiable on (a,b), and $g'(x)=f(x)$.

$$g'(x)=\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example 2

(a) $y=\int_a^x (t^2+t+1)dt$, $\frac{dy}{dx}=\frac{d}{dx}\int_a^x (t^2+t+1)dt = x^2+x+1$

(b) $y=\int_a^x te^t dt$, $\frac{dy}{dx}=\frac{d}{dx}\int_a^x te^t dt = xe^x$

(c) $y=\int_x^4 te^t dt$, $\frac{dy}{dx}=\frac{d}{dx}\int_x^4 te^t dt = \frac{d}{dx}\left(-\int_4^x te^t dt\right)=-\frac{d}{dx}\left(\int_4^x te^t dt\right)=-xe^x$

(d) $y=\int_0^{x^2}\sin t\, dt$

Let $u=x^2$. Then $y=\int_0^u \sin t\, dt$.

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{d}{du}\left(\int_0^u \sin t\, dt\right)\bullet\frac{du}{dx}=\sin u\bullet 2x = 2x\sin(x^2)$$

The Fundamental Theorem of Calculus, Part 2 (Integral Evaluation Theorem)

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b)-F(a)$$

The expression $F(b)-F(a)$ can be written as $F(x)\big]_a^b$, $\left[F(x)\right]_a^b$ or $F(x)\big|_a^b$ 。

Part 2 of The Fundamental Theorem of Calculus makes it possible for us to evaluate a definite integral (which is a limit of Riemann sums) in a much easier way, i.e., just by evaluating an antiderivative (indefinite integral) at the upper and lower limits of integration and taking the difference.

Example 3:

Use the Fundamental Theorem of Calculus to evaluate the following definite integrals

a) $\int_{-3}^{-1}\left(\frac{1}{x^2}-\frac{1}{x^3}\right)dx$ b) $\int_6^{10}\frac{1}{x+2}dx$ c) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\sin x dx$ d) $\int_4^{10}\frac{x+1}{x^2-2x-3}dx$

Solution (a):

$$\int_{-3}^{-1}\left(\frac{1}{x^2}-\frac{1}{x^3}\right)dx=\left[\frac{-1}{x}+\frac{1}{2x^2}\right]_{-3}^{-1}=\left(1+\frac{1}{2}\right)-\left(\frac{1}{3}+\frac{1}{18}\right)=\frac{10}{9}$$

2. SUBSTITUTION IN DEFINITE INTEGRALS

Theorem
If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$,
then $$\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

With the substitution $u = g(x)$, we have $du = g'(x)dx$ and transform the original integral involving *x* into one involving *u*. The new definite integral involving *u* has $g(a)$ and $g(b)$ as the corresponding new limits of integration.

<u>Transforming the integrand</u> and **<u>transforming the limits of integration</u>** **must be done <u>simultaneously</u>**.

If $F(u)$ is an antiderivative of $f(u)$ then $\int_{g(a)}^{g(b)} f(u)du = F(u)\Big]_{g(a)}^{g(b)}$.

<u>Example 4:</u>
Evaluate $\int_{-1}^{1} x^2\sqrt{x^3+1}\, dx$.

Solution:

Let $u = x^3 + 1$. Then $du = 3x^2 dx$, $x^2 dx = \frac{1}{3}du$

When $x = -1, u = 0$
$x = 1, u = 2$ [***Transforming limits of integration***.]

$$\therefore \int_{-1}^{1} x^2\sqrt{x^3+1}\, dx = \int_0^2 \sqrt{u} \cdot \frac{1}{3}du = \frac{1}{3}\int_0^2 u^{\frac{1}{2}}\, du = \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2 = \frac{2}{9}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = \frac{4\sqrt{2}}{9}$$

Alternatively,

Let $u = x^3 + 1$. Then $du = 3x^2 dx$, $x^2 dx = \frac{1}{3}du$

$$\int x^2\sqrt{x^3+1}\, dx = \int \sqrt{u} \cdot \frac{1}{3}du = \frac{1}{3}\int u^{\frac{1}{2}}\, du = \frac{1}{3}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + C = \frac{2}{9}(x^3+1)^{\frac{3}{2}} + C$$

$$\int_{-1}^{1} x^2\sqrt{x^3+1}\, dx = \left[\frac{2}{9}(x^3+1)^{\frac{3}{2}}\right]_{-1}^{1} = \frac{2}{9}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = \frac{4\sqrt{2}}{9}$$

3. INTEGRATION BY PARTS FOR DEFINITE INTEGRALS

Recalling the formula for integration by parts $\int udv = uv - \int vdu$
and combining with Part 2 of the Fundamental Theorem of Calculus,
we can evaluate definite integrals by parts using the following form.

$$\int_a^b udv == [uv]_a^b - \int_a^b vdu$$

Example 5

(a) $\int_0^{\frac{\pi}{4}} x \sin x \, dx$ (b) $\int_1^e x \ln x \, dx$

$u = x, \quad dv = \sin x dx$
$du = dx, \quad v = -\cos x$ **[←** *Same steps as for indefinite integral.***]**

$$\therefore \int_0^{\frac{\pi}{4}} x \sin x \, dx = \left[-x\cos x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\cos x \, dx = \left[-\frac{\pi}{4}\cos\frac{\pi}{4} - 0\right] + \int_0^{\frac{\pi}{4}} \cos x \, dx$$

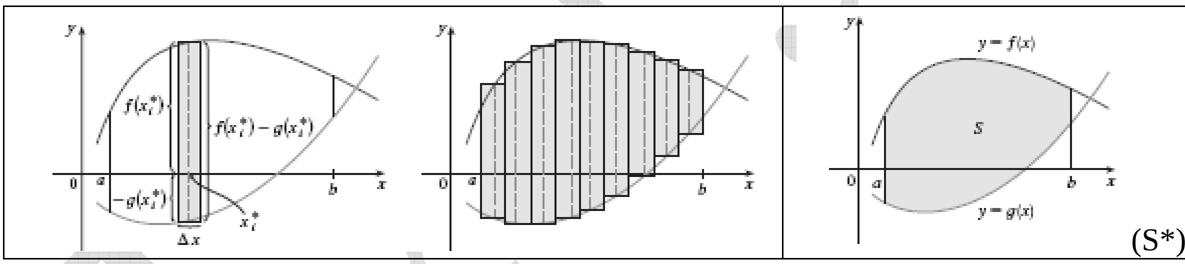
$$= -\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \left[\sin x\right]_0^{\frac{\pi}{4}} = -\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \left[\sin\frac{\pi}{4} - 0\right] = -\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

4. AREA BETWEEN TWO GIVEN CURVES

Area between Curves

If f and g are continuous functions with $f(x) \geq g(x)$ for all x in $[a,b]$, then the area A of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is given by $A = \int_a^b [f(x) - g(x)]dx$

[Here the graph of f is above the graph of g.]



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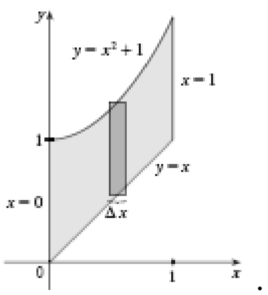
Example 6

Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$. (S*)

Solution

The region is shown in the figure



The area is given by

$$\int_0^1 [(x^2+1) - x]dx = \int_0^1 (x^2 - x + 1)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^1$$

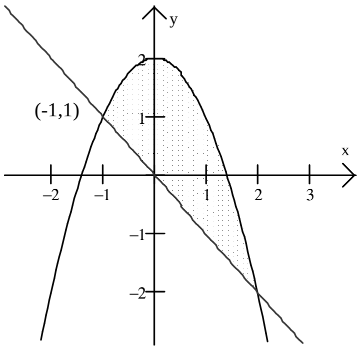
$$= \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}$$

<u>Example 7:</u>

Find the area of the region bounded by $y = 2 - x^2$ and $y = -x$.
(In this example, the region is determined just by the two graphs, no vertical lines are given.)

Solution:

We sketch the two curves.



To get the limits of integration, look for the points of intersection:

$$2 - x^2 = -x$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1, 2$$

The region runs from $x = -1$ to $x = 2$, providing us the limits of integration.

$$\therefore A = \int_{-1}^{2}\left[\left(2 - x^2\right) - (-x)\right]dx = \int_{-1}^{2}\left(2 + x - x^2\right)dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

<u>Special case:</u>

<u>Area bounded by the curve $y = f(x)$ and the *x*-axis on the interval [a, b].</u>

(a) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx$ is the area of the region bounded by the curve $y = f(x)$ and the *x*-axis between the vertical lines $x = a$ and $x = b$.

(b) If $f(x) \leq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx$ is the negative of the area of the region bounded by the curve $y = f(x)$ and the *x*-axis between the vertical lines $x = a$ and $x = b$.

(c) When f is not always positive or not always negative on $[a, b]$, the area between the graph of $y = f(x)$ and the *x*-axis over the interval $[a, b]$ can be obtained through these steps.

i) partition $[a, b]$ with the zeros of f [To find zeros of *f*, solve $f(x) = 0$]
ii) integrate f over each subinterval
iii) add the absolute values of the integrals

<u>Example 8:</u>

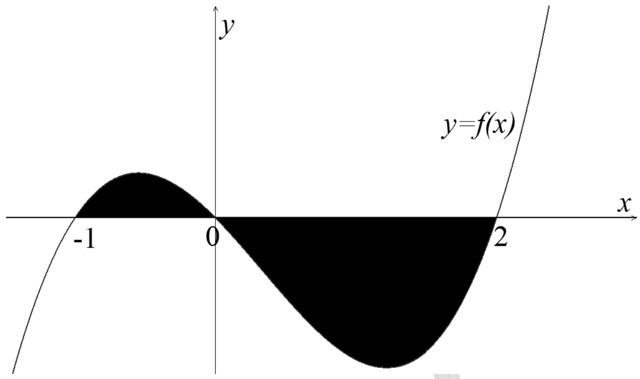
Find the area of the region between the *x*-axis and the graph of $f(x) = x^3 - x^2 - 2x$, for $-1 \leq x \leq 2$

Solution:

$$f(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$$

The zeros are x = 0, -1 and 2. The zeros partition [-1,2] into two subintervals [-1,0] and [0,2]

A sketch of the graph:

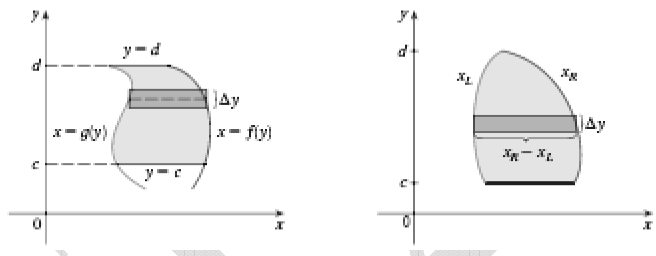


$$\int_{-1}^{0}\left(x^3-x^2-2x\right)dx=\left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{-1}^{0}=0-\left(\frac{1}{4}+\frac{1}{3}-1\right)=\frac{5}{12}$$

$$\int_{0}^{2}\left(x^3-x^2-2x\right)dx=\left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{0}^{2}=\left(4-\frac{8}{3}-4\right)=-\frac{8}{3}$$

$$\therefore \text{Total enclosed area}=\frac{5}{12}+\left|-\frac{8}{3}\right|=\frac{37}{12}$$

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x=f(y)$, $x=g(y)$, $y=c$ and $y=d$, where f and g are continuous and $f(y)\geq g(y)$ for all y in $[c,d]$, then its area is $A=\int_c^d\left[f(y)-g(y)\right]dy$



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Other Similar APPLICATIONS OF DEFINITE INTEGRALS

(a) Volume of Solid of Revolution
(b) Length of Arc, and
(c) Area of a Surface of Revolution

All these involve getting the results in the form of definite integrals. The same integration techniques are involved.

5. IMPROPER INTEGRALS

Improper integrals would be useful when you study convergence of infinite series (in this course) and when you study probability (not in this course).

Improper Integrals with Infinite Integration Limits

1) If $f(x)$ is continuous on $[a,\infty)$, then $\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$

2) If $f(x)$ is continuous on $(-\infty,b]$, then $\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$

3) If $f(x)$ is continuous on $(-\infty,\infty)$, then $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$ where c is any real number

Remarks:

1. If the limit is finite, the improper integral **converges** and the limit is the value of the improper integral. The improper integral is said to be **convergent**.
2. If the limit fails to exist, the improper integral **diverges;** it is said to be **divergent.**

Example 7:

Consider the improper integral $\int_1^{\infty} \frac{\ln x}{x^2}dx$

Solution:

$$\int_1^b \frac{\ln x}{x^2}dx = \left[-\frac{1}{x}\ln x\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx \qquad \text{using integration by parts: } u=\ln x$$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b \quad = -\frac{\ln b}{b} - \frac{1}{b} + 1$$

$$\int_1^{\infty} \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

$$= -\lim_{b\to\infty}\frac{\ln b}{b} - 0 + 1 = -\lim_{b\to\infty}\frac{\frac{1}{b}}{1} + 1 \; = 0 + 1 = 1 \qquad \text{[l'Hopital's rule]}$$

The improper integral $\int_1^{\infty} \frac{\ln x}{x^2}dx$ is convergent and $\int_1^{\infty} \frac{\ln x}{x^2}dx = 1$

Improper Integrals with Infinite Discontinuities

1) If $f(x)$ is continuous on $(a,b]$, then $\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$

2) If $f(x)$ is continuous on $[a,b)$, then $\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$

3) If $f(x)$ is continuous on $[a,c)\cup(c,b]$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Example 8:

Consider the improper integral $\int_0^1 \frac{1}{1-x}dx$

Solution:

The integrand $f(x) = \frac{1}{1-x}$ is continuous on $[0,1)$ but becomes infinite as $x \to 1^-$.

We evaluate the integral as $\lim_{b\to 1^-}\int_0^b \frac{1}{1-x}dx = \lim_{b\to 1^-}\left[-\ln|1-x|\right]_0^b = \lim_{b\to 1^-}\left[-\ln(1-b)+0\right] = \infty$

The improper integral $\int_0^1 \frac{1}{1-x}dx$ is divergent.

(nby, Nov 2015)